14.6 Directional Derivatives. Gradient Vector. Tangent Plane.

Consider $z = f(x,y)$ defined in a domain $D \subset \mathbb{R}^2$

Partial Derivatives at $(x,y) = (x_0, y_0)$.

a) $$\frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$ if limit exists.

b) $$\frac{\partial f}{\partial y}(x_0,y_0) = f_y(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$ if limit exists.

Directional Derivatives at $(x,y) = (x_0,y_0)$

For this, we need to choose a direction $\vec{u} = \langle a,b \rangle$, where $\|\vec{u}\| = 1$.

Then, directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of $\vec{u} = \langle a,b \rangle$ is defined as

$$D_u f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$ if this limit exists.

Theorem 3 (in book)

If f is a differentiable function of x and y, then f has directional derivatives in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and

$$\boxed{D_u f(x,y) = f_x(x,y)a + f_y(x,y)b}$$

Definition of Gradient.

If $f(x,y)$ has partial derivatives on D, the gradient of f is the vector function ∇f defined by

$$\boxed{\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle}$$

Corollary of theorem 3

If f is a differentiable function of x and y in D, then the directional derivative of f in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ can be written as

$$\boxed{D_u f(x,y) = \nabla f(x,y) \cdot \vec{u}}$$

<u>Exercise</u>: Given $\boxed{f(x,y) = \sin(2x+3y)}$

Find directional derivative in the direction of $\vec{u} = \langle \sqrt{3}, -1 \rangle$ at $(x_0, y_0) = (-6, 4)$.

<u>Answ</u>:

$$\nabla f = \langle f_x, f_y \rangle = \langle 2\cos(2x+3y), 3\cos(2x+3y) \rangle$$

Then, $\hat{u} = \frac{1}{2}\langle \sqrt{3}, -1 \rangle$ (unit vector)

$$D_{\hat{u}} f(-6,4) = \langle 2\cos(\underbrace{2(-6)+3(4)}_{=0}), 3\cos(\underbrace{2(-6)+3(4)}_{=0}) \rangle \cdot \frac{1}{2}\langle \sqrt{3}, -1 \rangle$$

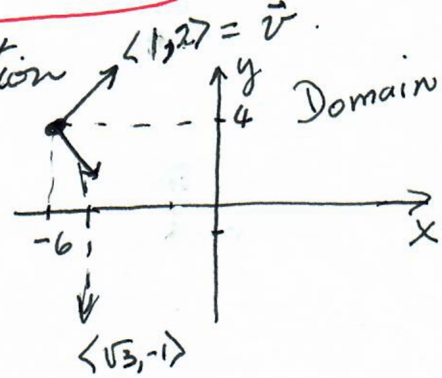
or

$$D_{\hat{u}} f(-6,4) = \frac{1}{2}\langle 2, 3 \rangle \cdot \langle \sqrt{3}, -1 \rangle = \frac{1}{2}(2\sqrt{3}-3)$$

$$= \sqrt{3} - \frac{3}{2}.$$

or $\boxed{D_{\hat{u}} f(-6,4) \simeq 0.2321}$

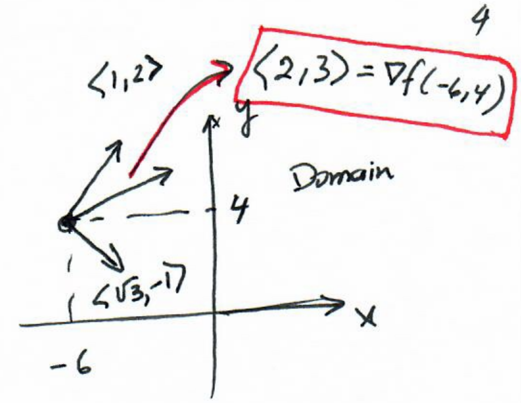
Let's another direction



$$\hat{v} = \frac{1}{\sqrt{5}}\langle 1, 2 \rangle$$

$$\boxed{D_{\hat{v}} f(-6,4) = \frac{1}{\sqrt{5}}\langle 2,3 \rangle \cdot \langle 1, 2 \rangle = \frac{8}{\sqrt{5}} \simeq 3.57}$$

In the direction of the gradient



$$\vec{w} = \nabla f(-6,4) = \langle 2,3 \rangle$$

$$\hat{w} = \frac{1}{\sqrt{13}} \langle 2,3 \rangle$$

$$\boxed{D_{\hat{w}} f(-6,4) = \frac{1}{\sqrt{13}} \langle 2,3 \rangle \cdot \langle 2,3 \rangle = \frac{13}{\sqrt{13}} = \sqrt{13} \simeq 3.6056}$$

The directional derivative is highest in the direction of the gradient.

<u>Theorem 15 (in book)</u>

If f is a differentiable function of 2 or 3 variables then, the maximum value of $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\vec{x})$.

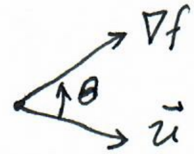
<u>Proof</u>. From Corollary of theorem 3

$$D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} = |\nabla f(\vec{x})| \overbrace{|\vec{u}|}^{=1} \cos\theta, \quad 0 < \theta < \pi$$

$$= |\nabla f(\vec{x})| \cos\theta$$

This quantity reaches a maximum when $\theta = 0 \Rightarrow \vec{u} = \nabla f(\vec{x})$ and it reaches a minimum when $\theta = \pi \Rightarrow \vec{u} = -\nabla f(\vec{x})$

In our previous exercise



i) max rate of change $= |\nabla f(-6,4)| = |\langle 2,3\rangle|$
$\theta = 0$
$= \sqrt{13}$

$f(x,y)$ increases fastest in the direction of the gradient $= \langle 2,3\rangle$, when $\theta = 0$

ii) min rate of change $= -|\nabla f(-6,4)| = -\sqrt{13}$

$f(x,y)$ decreases fastest in the direction: $-\nabla f(-6,4) = -\langle 2,3\rangle$
$\theta = \pi$

The above concepts be apply to solve problems: 27 − 36 in Section 14.6.

Theorem.

If $f(x,y,z)$ is a differentiable function of three Variables in a domain D in $\mathbb{R}^3$. and

i) S is a level Surface with equation

$$\boxed{S:\ f(x,y,z) = K.}$$

ii) $P_0\ (x_0, y_0, z_0)$ is a point on S.

iii) $\mathcal{C}$ is a curve (arbitrary) that lies on the Surface S and passes through P_0

iv) $\mathcal{C}$ is parametrically represented by a differentiable function

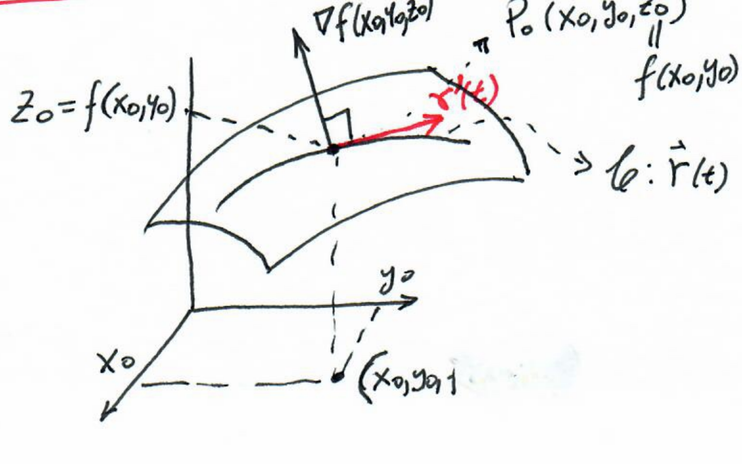
$$\boxed{\mathcal{C}:\ \vec{r}(t) = \langle x(t), y(t), z(t) \rangle}$$

And the point P_0 Corresponds to the vector

$$\boxed{\vec{r}(t_0) = \langle x(t_0), y(t_0), z(t_0) \rangle}$$

then,

$$\boxed{\nabla f(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0}$$



Proof: Consider the

Level Surface $S: f(x,y,z)=K$

and $\mathcal{C}$ a curve lying in S, so

$$\boxed{\mathcal{C}: \vec{r}(t) = \langle x(t), y(t), z(t)\rangle} \qquad (7.1)$$

Which passes through $P_0(x_0, y_0, z_0)$ in S.

Therefore, there is t_0 in $\mathbb{R}$, such that

$$\vec{r}(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle$$

Also, all points $P(x(t), y(t), z(t))$ in $\mathcal{C}$ are also in S, then they satisfy the equation for the level Surface S:

$$\boxed{f(x(t), y(t), z(t)) = K, \text{ for all } t} \qquad (7.2)$$

Differentiating this equation with respect to t. and applying the chain rule leads to

$$\frac{d}{dt}\left[f(x(t), y(t), z(t))\right] = \frac{d}{dt}(K) = 0$$

or $$\frac{\partial f}{\partial x}(\,)\frac{dx}{dt} + \frac{\partial f}{\partial y}(\,)\frac{dy}{dt} + \frac{\partial f}{\partial z}(\,)\frac{dz}{dt} = 0$$

Which can be written as

$$\nabla f(x(t), y(t), z(t)) \cdot \left\langle \frac{dx}{dt}(t), \frac{dy}{dt}(t), \frac{dz}{dt}(t) \right\rangle = 0,$$

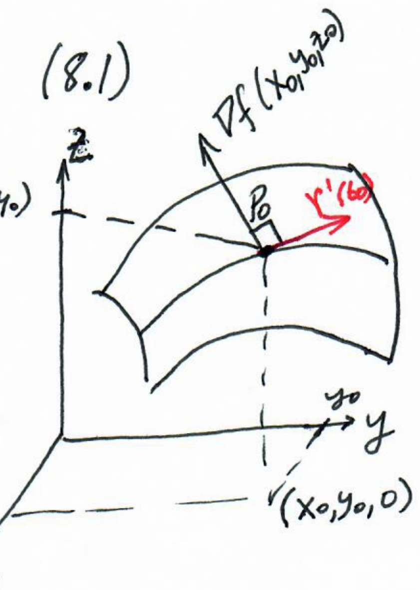
for all t

In particular, this is true for $t = t_0$

which leads to

$$\nabla f(x(t_0), y(t_0), z(t_0)) \cdot \langle x'(t_0), y'(t_0), z'(t_0) \rangle = 0$$

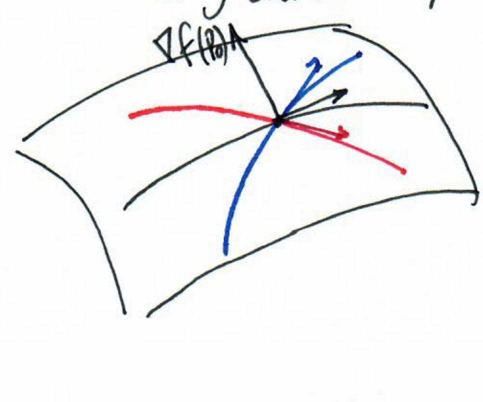
or $$\boxed{\nabla f(\;) \cdot \vec{r}\,'(t_0) = 0}. \quad (8.1)$$



This means that the gradient $\nabla f(x_0,y_0,z_0) \perp$ tangent vector $\vec{r}\,'(t_0)$

Because (8.1) is true for any $\vec{r}(t)$ describing an arbitrary curve $\mathcal{C}$ passing through P_0, then

$\nabla f(x_0,y_0,z_0) \perp$ tangent vector of any curve $\mathcal{C}$ passing through P_0 at P_0.



Definition (Tangent plane)

Assume $f(x,y,z)$ defined on a domain D is differentiable and $\nabla f(x_0,y_0,z_0) \neq 0$, for (x_0,y_0,z_0) in D.

then, the tangent plane to the level surface

$$S: f(x,y,z) = K, \text{ passing throug } P_0(x_0,y_0,z_0)$$

is defined as the plane π with normal vector $\vec{n} = \nabla f(x_0,y_0,z_0)$.

It means

$$\boxed{\pi: \nabla f(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0} \quad (9.1)$$

Which is equivalent to:

$$\boxed{f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0} \quad (9.2)$$

Definition (Normal line to S at P_0)

The line passing through P_0 and perpendicular to the tangent plane of S at P_0 is called the normal line to S at P_0.

Exercise (14.6 Sed. 14.6):

Find tangent plane and normal line to the Surface

$$S:\ x+y+z=e^{xyz},\quad \text{at } P_0(0,0,1).$$

Answ: $S: f(x,y,z) = x+y+z-e^{xyz}=0$ Level Surface

Therefore, $\nabla f(x,y,z) = \langle 1-yz\,e^{xyz},\ 1-xz\,e^{xyz},\ 1-xy\,e^{xyz}\rangle$

Then,

$$\boxed{\nabla f(0,0,1) = \langle 1,1,1\rangle}$$

and the tangent plane is given by the equation

$$\langle 1,1,1\rangle \cdot \langle x-0,\ y-0,\ z-1\rangle = 0$$

$$\boxed{x+y+z=1}$$

Also, the normal line is given by

$$\langle x,y,z\rangle = t\langle 1,1,1\rangle + \langle 0,0,1\rangle$$

or $$\boxed{\langle x,y,z\rangle = \langle t,t,t+1\rangle}.$$